A Clifford algebraic approach to reflection groups and root systems

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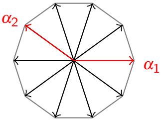
Reflection groups: a new approach

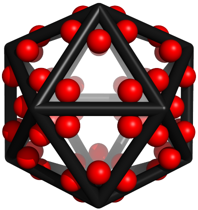


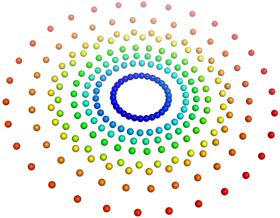
$$s_\alpha(v) = v - 2\frac{(v|\alpha)}{(\alpha|\alpha)}\alpha = -\alpha v \alpha = -(-\alpha)v(-\alpha)$$

- Work at the level of root systems (which define reflection groups)
- Interested in non-crystallographic root systems e.g. viruses, fullerenes etc. But: no Lie algebra, so conventionally less studied
- Clifford algebra is a uniquely suitable framework for reflection groups/root systems: reflection formula, spinor double covers, complex/quaternionic quantities arising as geometric objects

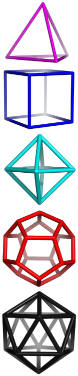
Main results

- Framework for reflection, conformal, modular and braid groups
- New view on the geometry of the Coxeter plane
- Induction of exceptional root systems and ADE from Platonic symmetries
- Naturally defines a range of representations





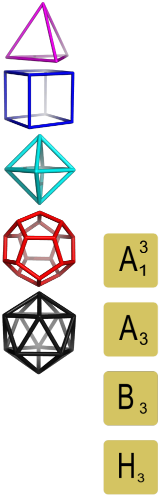
Platonic Solids



Platonic Solid	Group	root system
Tetrahedron	A_3	Cuboctahedron
	A_1^3	Octahedron
Octahedron	B_3	Cuboctahedron
Cube		+Octahedron
Icosahedron	H_3	Icosidodecahedron
Dodecahedron		

- Platonic Solids have been known for millennia

Platonic Solids

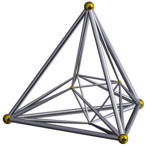
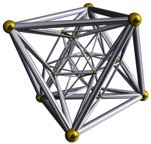
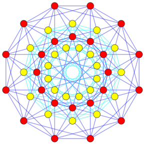
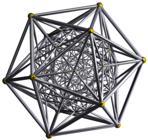


Platonic Solid	Group	root system
Tetrahedron	A_3 A_1^3	Cuboctahedron Octahedron
Octahedron Cube	B_3	Cuboctahedron +Octahedron
Icosahedron Dodecahedron	H_3	Icosidodecahedron

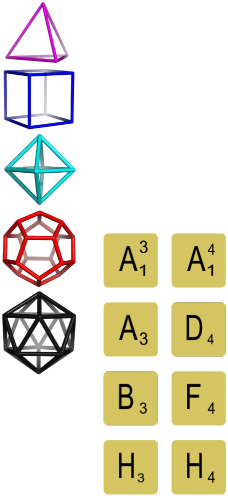
- Platonic Solids have been known for millennia
- Described by Coxeter groups

4D analogues of the Platonic Solids

- The 16-cell, 24-cell, 24-cell and dual 24-cell, the 600-cell and the 120-cell
- In higher dimensions there are only hypersimplices and hypercubes/octahedra (A_n and B_n)

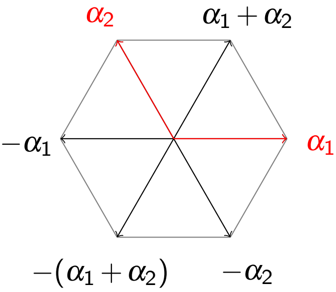





Platonic Solids



- Abundance of 4D root systems – exceptional
- Concatenating 3D reflections gives 4D Clifford spinors (binary polyhedral groups)
- These induce 4D root systems
  $R = a_0 + a_1 e_2 e_3 + a_2 e_3 e_1 + a_3 e_1 e_2 \Rightarrow$
  $R\tilde{R} = a_0^2 + a_1^2 + a_2^2 + a_3^2$
- This construction accidental to 3D perhaps explains the unusual abundance of 4D root systems

Root systems



Root system Φ: set of vectors α in a vector space with an inner product such that

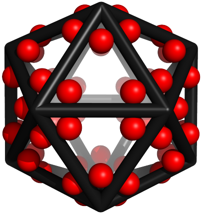
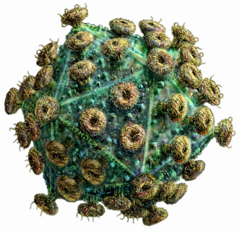
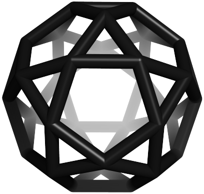
1. $\boxed{\Phi \cap \mathbb{R}\alpha = \{-\alpha, \alpha\} \ \forall\, \alpha \in \Phi}$
2. $\boxed{s_\alpha \Phi = \Phi \ \forall\, \alpha \in \Phi}$

reflection groups

$$\boxed{s_\alpha : v \to s_\alpha(v) = v - 2\frac{(v|\alpha)}{(\alpha|\alpha)}\alpha}$$

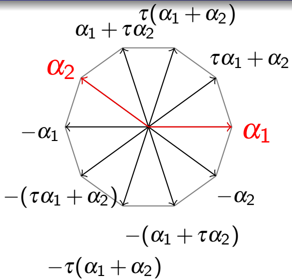
Simple roots: express every element of Φ via a $\mathbb{Z}$-linear combination with coefficients of the same sign.

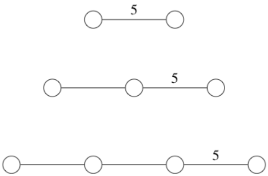
The Icosahedron

- Rotational icosahedral group is $I = A_5$ of order 60
- Full icosahedral group is H_3 of order 120 (including reflections/inversion); generated by the root system icosidodecahedron

Non-crystallographic Coxeter groups $H_2 \subset H_3 \subset H_4$





$$A=\begin{pmatrix}2 & -\tau\\ -\tau & 2\end{pmatrix}$$

$$A=\begin{pmatrix}2 & -1 & 0\\ -1 & 2 & -\tau\\ 0 & -\tau & 2\end{pmatrix}$$

$$A=\begin{pmatrix}2 & -1 & 0 & 0\\ -1 & 2 & -1 & 0\\ 0 & -1 & 2 & -\tau\\ 0 & 0 & -\tau & 2\end{pmatrix}$$

$H_2 \subset H_3 \subset H_4$: 10, 120, 14,400 elements, the only Coxeter groups that generate rotational symmetries of order 5.

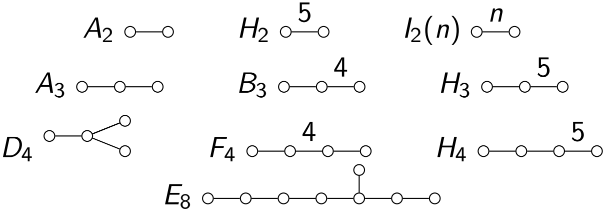
Linear combinations now in the extended integer ring

$\boxed{\mathbb{Z}[\tau]=\{a+\tau b \mid a,b\in\mathbb{Z}\}}$ golden ratio $\boxed{\tau=\frac{1}{2}(1+\sqrt{5})=2\cos\frac{\pi}{5}}$

$\boxed{x^2=x+1}$ $\boxed{\tau'=\sigma=\frac{1}{2}(1-\sqrt{5})=2\cos\frac{2\pi}{5}}$ $\boxed{\tau+\sigma=1, \tau\sigma=-1}$

Cartan-Dynkin diagrams

Coxeter-Dynkin diagrams: node = simple root, no link = roots orthogonal i.e. angle $\frac{\pi}{2}$, simple link = roots at angle $\frac{\pi}{3}$, link with label m = angle $\frac{\pi}{m}$.



Overview

Clifford Algebra and orthogonal transformations

- Geometric Product for two vectors $\boxed{ab \equiv a\cdot b + a\wedge b}$
- Inner product is symmetric part $a\cdot b = \frac{1}{2}(ab+ba)$
- Reflecting a in n is given by $\boxed{a' = a - 2(a\cdot n)n = -nan}$ (n and $-n$ doubly cover the same reflection)
- Via Cartan-Dieudonné theorem any orthogonal transformation can be written as successive reflections, which are doubly covered by Clifford versors/pinors A

$$\boxed{x' = \pm n_1 n_2 \dots n_k x n_k \dots n_2 n_1} =: \pm A x \tilde{A}$$

Clifford Algebra of 3D: the relation with 4D and 8D

- Clifford (Pauli) algebra in 3D is

$$\underbrace{\{1\}}_{\text{1 scalar}} \quad \underbrace{\{e_1, e_2, e_3\}}_{\text{3 vectors}} \quad \underbrace{\{e_1e_2, e_2e_3, e_3e_1\}}_{\text{3 bivectors}} \quad \underbrace{\{I \equiv e_1e_2e_3\}}_{\text{1 trivector}}$$

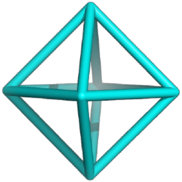
- We can multiply together root vectors in this algebra $\alpha_i\alpha_j \ldots$
- A general element has 8 components: 8D
- even products (rotations/spinors) have four components:

$$\boxed{R = a_0 + a_1e_2e_3 + a_2e_3e_1 + a_3e_1e_2 \Rightarrow R\tilde{R} = a_0^2 + a_1^2 + a_2^2 + a_3^2}$$

- So behaves as a 4D Euclidean object – inner product

$$\boxed{(R_1, R_2) = \frac{1}{2}(R_2\tilde{R}_1 + R_1\tilde{R}_2)}$$

Spinors from reflections: easy example



- The 6 roots $(\pm 1, 0, 0)$ and permutations in $A_1 \times A_1 \times A_1$
- $\boxed{\pm e_1,\ \pm e_2,\ \pm e_3}$ generate group of 8 spinors $\boxed{\pm 1, \pm e_1 e_2, \pm e_2 e_3, \pm e_3 e_1}$
- This is a discrete spinor group isomorphic to the quaternion group Q.

Pinors from reflections: easy example

$$\underbrace{\{\pm 1\}}_{\text{1 scalar}} \quad \underbrace{\{\pm e_1, \pm e_2, \pm e_3\}}_{\text{3 vectors}} \quad \underbrace{\{\pm e_1 e_2, \pm e_2 e_3, \pm e_3 e_1\}}_{\text{3 bivectors}} \quad \underbrace{\{\pm I \equiv e_1 e_2 e_3\}}_{\text{1 trivector}}$$

- The pin group also of course contains $\boxed{\pm e_1, \pm e_2, \pm e_3}$ and $\boxed{\pm e_1 e_2 e_3}$
- So total pin group is a group of order 16
- Since $\boxed{e_1, e_2, e_3}$ generate the inversion $e_1 e_2 e_3$, actually the 8 elements in the even subalgebra and the other 8 elements in the other 4D can be 'Hodge' dualised
- So when the group contains the inversion $\mathrm{Pin} = \mathrm{Spin} \times \mathbb{Z}_2$

Spinors from reflections: icosahedral case

- The H_3 root system has 30 roots e.g. simple roots $\boxed{\alpha_1 = e_2, \alpha_2 = -\frac{1}{2}((\tau-1)e_1 + e_2 + \tau e_3) \text{ and } \alpha_3 = e_3}$.
- Subgroup of rotations A_5 of order 60 is doubly covered by 120 spinors of the form $\boxed{\alpha_1\alpha_2 = -\frac{1}{2}(1-(\tau-1)e_1e_2 + \tau e_2e_3)}$, $\boxed{\alpha_1\alpha_3 = e_2e_3}$ and $\boxed{\alpha_2\alpha_3 = -\frac{1}{2}(\tau-(\tau-1)e_3e_1 + e_2e_3)}$.
- The inclusion of the H_3 inversion doubles this

Polyhedral groups as multivector groups

Group	Discrete subgroup	Order	Action Mechanism
$SO(3)$	rotational (chiral)	$\|G\|$	$x \to \tilde{R}xR$
$O(3)$	reflection (full/Coxeter)	$2\|G\|$	$x \to \pm\tilde{A}xA$
$\mathrm{Spin}(3)$	binary	$2\|G\|$	$(R_1, R_2) \to R_1R_2$
$\mathrm{Pin}(3)$	pinory (?)	$4\|G\|$	$(A_1, A_2) \to A_1A_2$

- e.g. the chiral icosahedral group has 60 elements, encoded by 120 spinors, which form the binary icosahedral group
- together with the inversion/pseudoscalar I this gives 60 rotations and 60 rotoinversions, i.e. the full icosahedral group H_3 in 120 elements doubly covered by 240 pinors

Some Group Theory: chiral, full, binary, pin

- Easy to calculate conjugacy classes etc
- Chiral (binary) polyhedral groups have irreps
- tetrahedral (12/24): $1, 1', 1'', 2_s, 2_s', 2_s'', 3$
- octahedral (24/48): $1, 1', 2, 2_s, 2_s', 3, 3', 4_s$
- icosahedral (60/120): $1, 2_s, 2_s', 3, \bar{3}, 4, 4_s, 5, 6_s$
- All binary are discrete subgroups of $SU(2)$ and all thus have a 2_s spinor irrep
- Connection with Trinities and the McKay correspondence

Tetrahedral group A_3: rotational group $\tilde{R}xR$

Simple roots for A_3:
$\alpha_1 = \frac{1}{\sqrt{2}}(e_2 - e_1)$, $\alpha_2 = \frac{1}{\sqrt{2}}(e_3 - e_2)$ and $\alpha_3 = \frac{1}{\sqrt{2}}(e_1 + e_2)$

Conj. Class	Distinct rotations given by two spinors each ($\pm$)
1	± 1
4	$\pm\frac{1}{2}(1 - e_1e_2 + e_2e_3 - e_3e_1)$, $\pm\frac{1}{2}(1 - e_1e_2 - e_2e_3 + e_3e_1)$,
	$\pm\frac{1}{2}(1 + e_1e_2 - e_2e_3 - e_3e_1)$, $\pm\frac{1}{2}(1 + e_1e_2 + e_2e_3 + e_3e_1)$
$\mathbf{4}^{-1}$	$\pm\frac{1}{2}(1 + e_1e_2 - e_2e_3 + e_3e_1)$, $\pm\frac{1}{2}(1 + e_1e_2 + e_2e_3 - e_3e_1)$,
	$\pm\frac{1}{2}(1 - e_1e_2 + e_2e_3 + e_3e_1)$, $\pm\frac{1}{2}(1 - e_1e_2 - e_2e_3 - e_3e_1)$
3	$\pm e_1e_2$, $\pm e_2e_3$, $\pm e_3e_1$

Tetrahedral group A_3: spinor group R_1R_2

Conjugacy Class	Group elements
$\mathbf{1}$	1
$\mathbf{1}_-$	-1
$\mathbf{4}$	$\frac{1}{2}(1-e_1e_2+e_2e_3-e_3e_1)$, $\frac{1}{2}(1-e_1e_2-e_2e_3+e_3e_1)$, $\frac{1}{2}(1+e_1e_2-e_2e_3-e_3e_1)$, $\frac{1}{2}(1+e_1e_2+e_2e_3+e_3e_1)$
$\mathbf{4}_-$	$-\frac{1}{2}(1-e_1e_2+e_2e_3-e_3e_1)$, $-\frac{1}{2}(1-e_1e_2-e_2e_3+e_3e_1)$, $-\frac{1}{2}(1+e_1e_2-e_2e_3-e_3e_1)$, $-\frac{1}{2}(1+e_1e_2+e_2e_3+e_3e_1)$
$\mathbf{4}^{-1}$	$\frac{1}{2}(1+e_1e_2-e_2e_3+e_3e_1)$, $\frac{1}{2}(1+e_1e_2+e_2e_3-e_3e_1)$, $\frac{1}{2}(1-e_1e_2+e_2e_3+e_3e_1)$, $\frac{1}{2}(1-e_1e_2-e_2e_3-e_3e_1)$
$\mathbf{4}_-^{-1}$	$-\frac{1}{2}(1+e_1e_2-e_2e_3+e_3e_1)$, $-\frac{1}{2}(1+e_1e_2+e_2e_3-e_3e_1)$, $-\frac{1}{2}(1-e_1e_2+e_2e_3+e_3e_1)$, $-\frac{1}{2}(1-e_1e_2-e_2e_3-e_3e_1)$
$\mathbf{6}$	$\pm e_1e_2$, $\pm e_2e_3$, $\pm e_3e_1$

Tetrahedral group A_3: pin group $A_1 A_2$

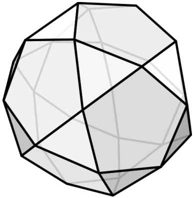
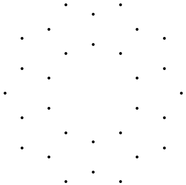
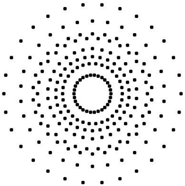
Conjugacy Class	Group elements
$\mathbf{1}$	1
$\mathbf{1}_-$	-1
$\mathbf{8}_+$	$\frac{1}{2}(1-e_1e_2+e_2e_3-e_3e_1)$, $\frac{1}{2}(1-e_1e_2-e_2e_3+e_3e_1)$, $\frac{1}{2}(1+e_1e_2-e_2e_3-e_3e_1)$, $\frac{1}{2}(1+e_1e_2+e_2e_3+e_3e_1)$, $\frac{1}{2}(1+e_1e_2-e_2e_3+e_3e_1)$, $\frac{1}{2}(1+e_1e_2+e_2e_3-e_3e_1)$, $\frac{1}{2}(1-e_1e_2+e_2e_3+e_3e_1)$, $\frac{1}{2}(1-e_1e_2-e_2e_3-e_3e_1)$
$\mathbf{8}_-$	$-\frac{1}{2}(1-e_1e_2+e_2e_3-e_3e_1)$, $-\frac{1}{2}(1-e_1e_2-e_2e_3+e_3e_1)$, $-\frac{1}{2}(1+e_1e_2-e_2e_3-e_3e_1)$, $-\frac{1}{2}(1+e_1e_2+e_2e_3+e_3e_1)$, $-\frac{1}{2}(1+e_1e_2-e_2e_3+e_3e_1)$, $-\frac{1}{2}(1+e_1e_2+e_2e_3-e_3e_1)$, $-\frac{1}{2}(1-e_1e_2+e_2e_3+e_3e_1)$, $-\frac{1}{2}(1-e_1e_2-e_2e_3-e_3e_1)$
$\mathbf{6}$	$\pm e_1e_2$, $\pm e_2e_3$, $\pm e_3e_1$
$\mathbf{12}$	$\frac{1}{\sqrt{2}}(\pm e_1 \pm e_2)$, $\frac{1}{\sqrt{2}}(\pm e_2 \pm e_3)$, $\frac{1}{\sqrt{2}}(\pm e_3 \pm e_1)$
$\mathbf{6}_+$	$\frac{1}{\sqrt{2}}(I \pm e_1)$, $\frac{1}{\sqrt{2}}(I \pm e_2)$, $\frac{1}{\sqrt{2}}(I \pm e_3)$
$\mathbf{6}_-$	$-\frac{1}{\sqrt{2}}(I \pm e_1)$, $-\frac{1}{\sqrt{2}}(I \pm e_2)$, $-\frac{1}{\sqrt{2}}(I \pm e_3)$

Doubly covers A_3.

Overview

The Coxeter Plane

- Every (for our purposes) Coxeter group has a Coxeter plane.
- A way to visualise Coxeter groups in any dimension by projecting their root system onto the Coxeter plane

Coxeter Elements, Degrees and Exponents

- Like the symmetric group, Coxeter groups can have invariant polynomials. Their degrees d are important invariants/group characteristics.
- Turns out that actually degrees d are intimately related to so-called exponents m $\boxed{m = d - 1}$.

Coxeter Elements, Degrees and Exponents

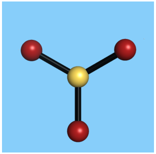
- A Coxeter Element is any combination of all the simple reflections $\boxed{w = s_1 \dots s_n}$, i.e. in Clifford algebra it is encoded by the versor $W = \alpha_1 \dots \alpha_n$ acting as $\boxed{v \to wv = \pm \tilde{W} v W}$. All such elements are conjugate and thus their order is invariant and called the Coxeter number h.
- The Coxeter element has complex eigenvalues of the form $\boxed{\exp(2\pi mi/h)}$ where m are called exponents:
  $\boxed{wx = \exp(2\pi mi/h)x}$
- Standard theory complexifies the real Coxeter group situation in order to find complex eigenvalues, then takes real sections again.

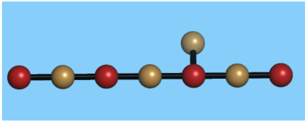
Coxeter Elements, Degrees and Exponents

- The Coxeter element has complex eigenvalues of the form $\boxed{\exp(2\pi mi/h)}$ where m are called exponents
- Standard theory complexifies the real Coxeter group situation in order to find complex eigenvalues, then takes real sections again.
- In particular, 1 and $h-1$ are always exponents
- Turns out that actually exponents and degrees are intimately related ($\boxed{m=d-1}$). The construction is slightly roundabout but uniform, and uses the Coxeter plane.

The Coxeter Plane

- In particular, can show every (for our purposes) Coxeter group has a Coxeter plane
- Existence relies on the fact that all groups in question have tree-like Dynkin diagrams, and thus admit an alternate colouring
- Essentially just gives two sets of mutually commuting generators





The Coxeter Plane

- Existence relies on the fact that all groups in question have tree-like Dynkin diagrams, and thus admit an alternate colouring
- Essentially just gives two sets of orthogonal = mutually commuting generators but anticommuting root vectors α_w and α_b (duals ω)
- Cartan matrices are positive definite, and thus have a Perron-Frobenius (all positive) eigenvector λ_i.
- Take linear combinations of components of this eigenvector as coefficients of two vectors from the orthogonal sets $v_w = \sum \lambda_w \omega_w$ and $v_b = \sum \lambda_b \omega_b$
- Their outer product/Coxeter plane bivector $\boxed{B_C = v_b \wedge v_w}$ describes an invariant plane where w acts by rotation by $2\pi/h$.

Clifford Algebra and the Coxeter Plane – 2D case

$$I_2(n) \qquad \circ\overset{n}{-\!\!-}\circ$$

- For $I_2(n)$ take $\boxed{\alpha_1 = e_1,\ \alpha_2 = -\cos\frac{\pi}{n}e_1 + \sin\frac{\pi}{n}e_2}$
- So Coxeter versor is just

$$\boxed{W = \alpha_1\alpha_2 = -\cos\frac{\pi}{n} + \sin\frac{\pi}{n}e_1e_2 = -\exp\left(-\frac{\pi I}{n}\right)}$$

- In Clifford algebra it is therefore immediately obvious that the action of the $I_2(n)$ Coxeter element is described by a versor (here a rotor/spinor) that encodes rotations in the e_1e_2-Coxeter-plane and yields $h = n$ since trivially $W^n = (-1)^{n+1}$ yielding $\boxed{w^n = 1}$ via $wv = \tilde{W}vW$.

Clifford Algebra and the Coxeter Plane – 2D case

- Coxeter versor $\boxed{W=-\cos\frac{\pi}{n}+\sin\frac{\pi}{n}e_1e_2=-\exp\left(-\frac{\pi I}{n}\right)}$
- $\boxed{I=e_1e_2}$ anticommutes with both e_1 and e_2 such that sandwiching formula becomes
  $\boxed{v\to wv=\tilde{W}vW=\tilde{W}^2v=\exp\left(\pm\frac{2\pi I}{n}\right)v}$ immediately
  yielding the standard result for the complex eigenvalues in real Clifford algebra without any need for artificial complexification
- The Coxeter plane bivector $B_C=e_1e_2=I$ gives the complex structure
- The Coxeter plane bivector B_C is invariant under the Coxeter versor $\tilde{W}B_CW=\pm B_C$.

Clifford algebra: no need for complexification

- Turns out in Clifford algebra we can factorise W into orthogonal (commuting/anticommuting) components $\boxed{W = \alpha_1 \dots \alpha_n = W_1 \dots W_n}$ with $\boxed{W_i = \exp(\pi m_i I_i/h)}$
- Here, I_i is a bivector describing a plane with $\boxed{I_i^2 = -1}$
- For v orthogonal to the plane descrbed by I_i we have $\boxed{v \rightarrow \tilde{W}_i v W_i = \tilde{W}_i W_i v = v}$ so cancels out
- For v in the plane we have $\boxed{v \rightarrow \tilde{W}_i v W_i = \tilde{W}_i^2 v = \exp(2\pi m_i I_i/h) v}$
- Thus if we decompose W into orthogonal eigenspaces, in the eigenvector equation all orthogonal bits cancel out and one gets the complex eigenvalue from the respective eigenspace

Clifford algebra: no need for complexification

- For v in the plane we have
  $$\boxed{v \to \tilde{W}_i v W_i = \tilde{W}_i^2 v = \exp(2\pi m_i I_i/h) v}$$
- So complex eigenvalue equation arises geometrically without any need for complexification
- Different complex structures immediately give different eigenplanes
- Eigenvalues/angles/exponents given from just factorising $W = \alpha_1 \dots \alpha_n$
- E.g. H_4 has exponents $1, 11, 19, 29$ and $W = \exp\left(\frac{\pi}{30} B_C\right) \exp\left(\frac{11\pi}{30} I B_C\right)$
- Here we have been looking for orthogonal eigenspaces, so innocuous – different complex structures commute
- But not in general – naive complexification can be misleading

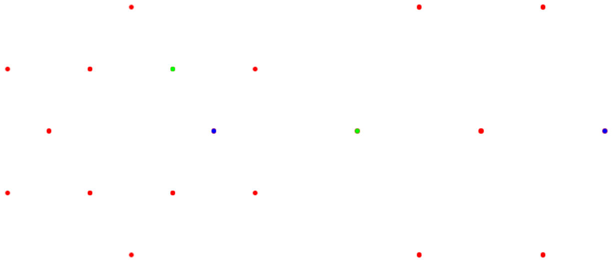
4D case: D_4

- E.g. D_4 has exponents $1, 3, 3, 5$
- Coxeter versor decomposes into orthogonal components

$$\boxed{W = \alpha_1\alpha_2\alpha_3\alpha_4 = e_1e_2e_3e_4 - e_2e_3 - e_1e_2 + e_1e_3}$$

$$\boxed{= \frac{1}{2}(\sqrt{3} - B_C)IB_C = \exp\left(\frac{\pi}{6}B_C\right)\exp\left(\frac{3\pi}{6}IB_C\right)}$$

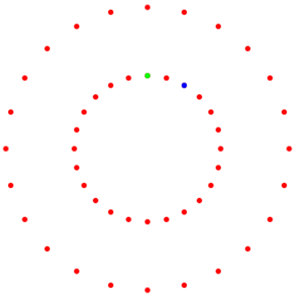
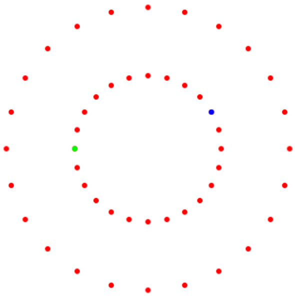
$$B_C = 1/\sqrt{3}(e_1 + e_2 + e_3)e_4;\ IB_C = (e_1 + e_2 - 2e_3)(e_1 - e_2)$$



4D case: F_4

- E.g. F_4 has exponents $1, 5, 7, 11$
- Coxeter versor decomposes into orthogonal components

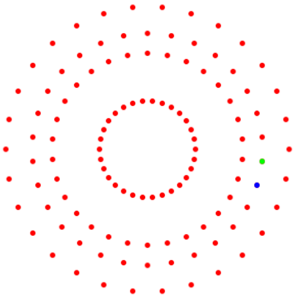
$$\boxed{W = \alpha_1\alpha_2\alpha_3\alpha_4 = \exp\left(\frac{\pi}{12}B_C\right)\exp\left(\frac{5\pi}{12}IB_C\right)}$$

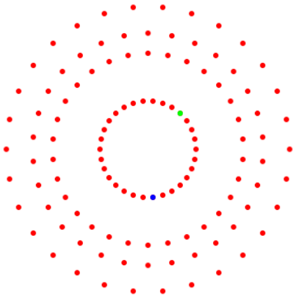



4D case: H_4

- E.g. H_4 has exponents $1, 11, 19, 29$
- Coxeter versor decomposes into orthogonal components

$$\boxed{W = \alpha_1\alpha_2\alpha_3\alpha_4 = \exp\left(\frac{\pi}{30}B_C\right)\exp\left(\frac{11\pi}{30}IB_C\right)}$$





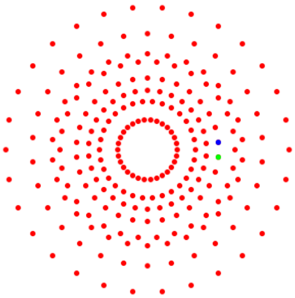
Clifford Algebra and the Coxeter Plane – 4D case summary

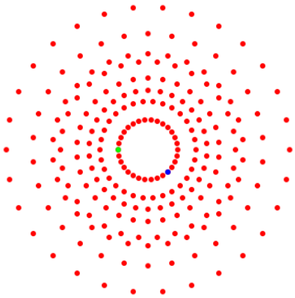
rank 4	exponents	W-factorisation
A_4	$1, 2, 3, 4$	$W = \exp\left(\frac{\pi}{5} B_C\right) \exp\left(\frac{2\pi}{5} IB_C\right)$
B_4	$1, 3, 5, 7$	$W = \exp\left(\frac{\pi}{8} B_C\right) \exp\left(\frac{3\pi}{8} IB_C\right)$
D_4	$1, 3, 3, 5$	$W = \exp\left(\frac{\pi}{6} B_C\right) \exp\left(\frac{\pi}{2} IB_C\right)$
F_4	$1, 5, 7, 11$	$W = \exp\left(\frac{\pi}{12} B_C\right) \exp\left(\frac{5\pi}{12} IB_C\right)$
H_4	$1, 11, 19, 29$	$W = \exp\left(\frac{\pi}{30} B_C\right) \exp\left(\frac{11\pi}{30} IB_C\right)$

8D case: E_8

- E.g. H_4 has exponents $1, 11, 19, 29$, E_8 has $1, 7, 11, 13, 17, 19, 23, 29$
- Coxeter versor decomposes into orthogonal components

$$\boxed{W = \alpha_1 \dots \alpha_8 = \exp\left(\frac{\pi}{30}B_C\right)\exp\left(\frac{7\pi}{30}B_2\right)\exp\left(\frac{11\pi}{30}B_3\right)\exp\left(\frac{13\pi}{30}B_4\right)}$$





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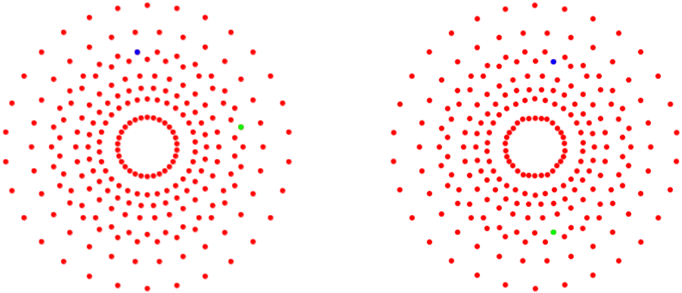
- E.g. H_4 has exponents $1, 11, 19, 29$, E_8 has $1, 7, 11, 13, 17, 19, 23, 29$
- Coxeter versor decomposes into orthogonal components

$$\boxed{W = \alpha_1 \dots \alpha_8 = \exp(\frac{\pi}{30} B_C) \exp(\frac{7\pi}{30} B_2) \exp(\frac{11\pi}{30} B_3) \exp(\frac{13\pi}{30} B_4)}$$



Imaginary differences – different imaginaries

So what has been gained by this Clifford view?

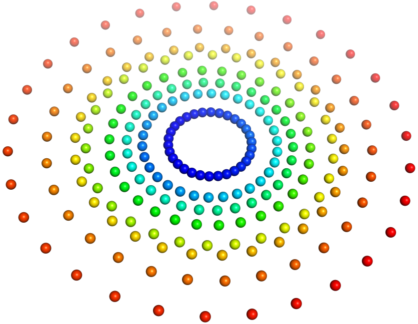
- There are different entities that serve as unit imaginaries
- They have a geometric interpretation as an eigenplane of the Coxeter element
- These don't need to commute with everything like i (though they do here – at least anticommute. But that is because we looked for orthogonal decompositions)
- But see that in general naive complexification can be a dangerous thing to do – unnecessary, issues of commutativity, confusing different imaginaries etc

Overview

1. Polyhedral groups, Platonic solids and root systems
2. Reflection groups with Clifford algebras
   - A Clifford way of doing orthogonal transformations
   - The geometry of the Coxeter plane
   - Root system induction and ADE correspondences
   - Representations from multivector groups
   - Conformal, modular and braid groups
3. Conclusions

Exceptional E_8 (projected into the Coxeter plane)

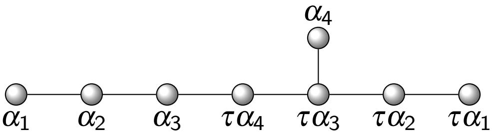
E_8 root system has 240 roots, H_3 has order 120

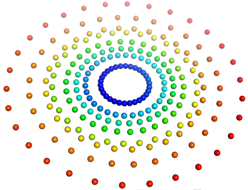


- Order 120 group H_3 doubly covered by 240 (s)pinors in 8D space

- With (somewhat counterintuitive) reduced inner product this gives the E_8 root system

- E_8 is actually hidden within 3D geometry!





Induction Theorem – root systems

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- Check axioms:
  1. $\boxed{\Phi \cap \mathbb{R}\alpha = \{-\alpha, \alpha\} \ \forall \, \alpha \in \Phi}$
  2. $\boxed{s_\alpha \Phi = \Phi \ \forall \, \alpha \in \Phi}$

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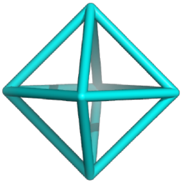
- In 2D, the space of spinors is also 2D and the root systems are self-dual under an analogous construction

Spinors from reflections: easy example



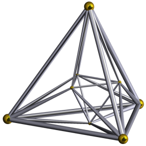
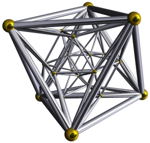
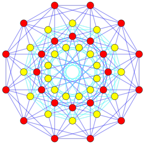
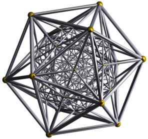
- The 6 roots $(\pm 1, 0, 0)$ and permutations in $A_1 \times A_1 \times A_1$ generate 8 spinors:
- $\boxed{\pm e_1,\ \pm e_2,\ \pm e_3}$ give the 8 spinors $\boxed{\pm 1, \pm e_1 e_2, \pm e_2 e_3, \pm e_3 e_1}$
- This is a discrete spinor group isomorphic to the quaternion group Q.
- As 4D vectors these are $(\pm 1, 0, 0, 0)$ and permutations, the 8 roots of $A_1 \times A_1 \times A_1 \times A_1$ (the 16-cell).

H_4 from H_3

- The H_3 root system has 30 roots e.g. simple roots $\boxed{\alpha_1 = e_2, \alpha_2 = -\frac{1}{2}((\tau-1)e_1 + e_2 + \tau e_3) \text{ and } \alpha_3 = e_3}$.
- Subgroup of rotations A_5 of order 60 is doubly covered by 120 spinors of the form $\boxed{\alpha_1\alpha_2 = -\frac{1}{2}(1-(\tau-1)e_1e_2 + \tau e_2e_3)}$, $\boxed{\alpha_1\alpha_3 = e_2e_3}$ and $\boxed{\alpha_2\alpha_3 = -\frac{1}{2}(\tau-(\tau-1)e_3e_1 + e_2e_3)}$.
- $$(\pm 1, 0, 0, 0) \text{ (8 perms) }, \frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1) \text{ (16 perms)}$$
  $$\frac{1}{2}(0, \pm 1, \pm\sigma, \pm\tau) \text{ (96 even perms)},$$
  As 4D vectors are the 120 roots of the H_4 root system.

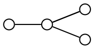
Spinors and Polytopes

- Can reinterpret spinors in $\mathbb{R}^3$ as vectors in $\mathbb{R}^4$
- Give (exceptional) root systems (D_4, F_4, H_4)
- They constitute the vertices of the 16-cell, 24-cell, 24-cell and dual 24-cell and the 600-cell
- These are 4D analogues of the Platonic Solids. Strange symmetries better understood in terms of 3D spinors

Trinity of 4D Exceptional Root Systems

- Exceptional phenomena: D_4 (triality, important in string theory), F_4 (largest lattice symmetry in 4D), H_4 (largest non-crystallographic symmetry); Exceptional D_4 and F_4 arise from series A_3 and B_3; $A_1 \times I_2(n) \rightarrow I_2(n) \times I_2(n)$

rank-3 group	diagram	binary	rank-4 group	diagram
$A_1 \times A_1 \times A_1$	$\circ \quad \circ \quad \circ$	Q	$A_1 \times A_1 \times A_1 \times A_1$	$\circ \quad \circ \quad \circ \quad \circ$
A_3	$\circ\!-\!\!-\!\circ\!-\!\!-\!\circ$	$2T$	D_4	
B_3	$\circ\!-\!\!-\!\circ\overset{4}{-\!\!-}\circ$	$2O$	F_4	$\circ\!-\!\!-\!\circ\overset{4}{-\!\!-}\circ\!-\!\!-\!\circ$
H_3	$\circ\!-\!\!-\!\circ\overset{5}{-\!\!-}\circ$	$2I$	H_4	$\circ\!-\!\!-\!\circ\!-\!\!-\!\circ\overset{5}{-\!\!-}\circ$

Arnold's indirect connection between Trinities (A_3, B_3, H_3) and (D_4, F_4, H_4)

- Arnold had noticed a handwavey connection:
- Decomposition of 3D groups in terms of number of Springer cones matches what are essentially the exponents of the 4D groups:
- A_3: $24 = 2(1+3+3+5)$ – D_4: $(1,3,3,5)$
- B_3: $48 = 2(1+5+7+11)$ – F_4: $(1,5,7,11)$
- H_3: $120 = 2(1+11+19+29)$ – H_4: $(1,11,19,29)$

Arnold's indirect connection between Trinities

rank 4	exponents	W-factorisation
A_4	$1,2,3,4$	$W=\exp\left(\frac{\pi}{5}B_C\right)\exp\left(\frac{2\pi}{5}IB_C\right)$
B_4	$1,3,5,7$	$W=\exp\left(\frac{\pi}{8}B_C\right)\exp\left(\frac{3\pi}{8}IB_C\right)$
D_4	$1,3,3,5$	$W=\exp\left(\frac{\pi}{6}B_C\right)\exp\left(\frac{\pi}{2}IB_C\right)$
F_4	$1,5,7,11$	$W=\exp\left(\frac{\pi}{12}B_C\right)\exp\left(\frac{5\pi}{12}IB_C\right)$
H_4	$1,11,19,29$	$W=\exp\left(\frac{\pi}{30}B_C\right)\exp\left(\frac{11\pi}{30}IB_C\right)$

The remaining cases in the root system induction construction work the same way, not just this Trinity! So more general correspondence:

$$(A_1\times I_2(n), A_3, B_3, H_3)\rightarrow (I_2(n)\times I_2(n), D_4, F_4, H_4)$$

The countably infinite family $I_2(n)$ and Arnold's construction

- For A_1^3 can see immediately $8 = 2(1+1+1+1)$
- Simple roots $\alpha_1 = e_1$, $\alpha_2 = e_2$, $\alpha_3 = e_3$, $\alpha_4 = e_4$ give $W = e_1e_2e_3e_4 = (\cos\frac{\pi}{2} + \sin\frac{\pi}{2}e_1e_2)(\cos\frac{\pi}{2} + \sin\frac{\pi}{2}e_3e_4) = \exp(\frac{\pi}{2}e_1e_2)\exp(\frac{\pi}{2}e_3e_4)$
- Gives exponents $(1,1,1,1)$ (from $h-1 = 2-1$)

The countably infinite family $I_2(n)$ and Arnold's construction

- For $A_1 \times I_2(n)$ one gets the same decomposition $4n = 2(1+(n-1)+1+(n-1)) = 2\cdot 2n$
- Simple roots $\alpha_1 = e_1$, $\alpha_2 = -\cos\frac{\pi}{n}e_1 + \sin\frac{\pi}{n}e_2$, $\alpha_3 = e_3$, $\alpha_4 = -\cos\frac{\pi}{n}e_3 + \sin\frac{\pi}{n}e_4$ give $W = \exp\left(-\frac{\pi e_1 e_2}{n}\right)\exp\left(-\frac{\pi e_3 e_4}{n}\right)$
- Gives exponents $(1,(n-1),1,(n-1))$

The countably infinite family $I_2(n)$ and Arnold's construction

- So Arnold's initial hunch regarding the exponents extends in fact to my full correspondence
- McKay correspondence is a correspondence between even subgroups of $SU(2)$/quaternions and ADE affine Lie algebras
- In fact here get the even quaternion subgroups from 3D – link to ADE affine Lie algebras via McKay?

3D, 4D and ADE correspondences

- McKay correspondence relates even $SU(2)$ subgroups with ADE Lie algebras $(A_{2n-1}, D_{n+2}, E_6, E_7, E_8)$
- Induction theorem: get these as 2D/4D root systems $(I_2(n)\times I_2(n), D_4, F_4, H_4)$ from 2D/3D root systems $A_1\times I_2(n), A_3, B_3, H_3)$
- $(2n+2, 12, 18, 30)$ are numbers of roots, the sum of the dimensions of the irreps and the ADE Coxeter number

4D	G	$\sum d_i$	ADE	h
$I_2(n)\times I_2(n)$	Dic_n	$2n+2$	$\tilde{A}_{2n-1}$	$2n$
			$\tilde{D}_{n+2}$	$2(n+1)$
D_4	$2T$	12	$\tilde{E}_6$	12
F_4	$2O$	18	$\tilde{E}_7$	18
H_4	$2I$	30	$\tilde{E}_8$	30

2D/3D, 2D/4D and ADE correspondences

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2D/3D	$\lvert\Phi\rvert$	4D	G	$\sum d_i$	ADE	h
$A_1 \times I_2(n)$	$2n+2$	$I_2(n) \times I_2(n)$	Dic$_n$	$2n+2$		
A_3	12	D_4	$2T$	12		
B_3	18	F_4	$2O$	18		
H_3	30	H_4	$2I$	30		

2D/3D, 2D/4D and ADE correspondences

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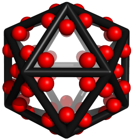
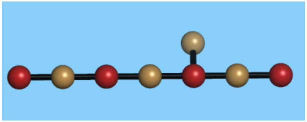
- $(2n, 2n+2, 12, 18, 30)$ are numbers of roots, the sum of the dimensions of the irreps and the ADE Coxeter number

2D/3D	$\|\Phi\|$	4D	G	$\sum d_i$	ADE	h
$I_2(n)$	$2n$	$I_2(n)$	C_{2n}	$2n$	$\tilde{A}_{2n-1}$	$2n$
$A_1 \times I_2(n)$	$2n+2$	$I_2(n) \times I_2(n)$	Dic_n	$2n+2$	$\tilde{D}_{n+2}$	$2(n+1)$
A_3	12	D_4	$2T$	12	$\tilde{E}_6$	12
B_3	18	F_4	$2O$	18	$\tilde{E}_7$	18
H_3	30	H_4	$2I$	30	$\tilde{E}_8$	30

Is there a direct Platonic-ADE correspondence?

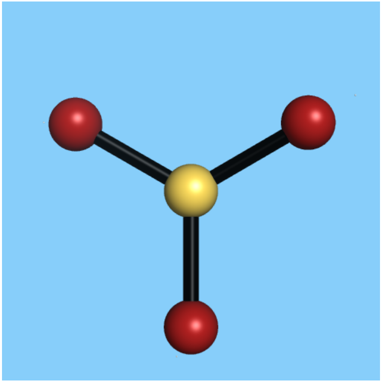




2D/3D		rot	ADE		legs
$I_2(n)$	n	n	A_n	$\cdots$	n
$A_1 \times I_2(n)$	n	$2, 2, n$	D_{n+2}	$\cdots$	$2, 2, n$
A_3		$2, 3, 3$	E_6		$2, 3, 3$
B_3	4	$2, 3, 4$	E_7		$2, 3, 4$
H_3	5	$2, 3, 5$	E_8		$2, 3, 5$

A Trinity of root system ADE correspondences

- 2D/3D root systems $(I_2(n), A_1 \times I_2(n), A_3, B_3, H_3)$
- 2D/4D root systems $(I_2(n), I_2(n) \times I_2(n), D_4, F_4, H_4)$
- ADE root systems $(A_n, D_{n+2}, E_6, E_7, E_8)$



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Polyhedral groups as multivector groups

Group	Discrete subgroup	Order	Action Mechanism
$SO(3)$	rotational (chiral)	$\|G\|$	$x \rightarrow \tilde{R}xR$
$O(3)$	reflection (full/Coxeter)	$2\|G\|$	$x \rightarrow \pm\tilde{A}xA$
$\mathrm{Spin}(3)$	binary	$2\|G\|$	$(R_1, R_2) \rightarrow R_1R_2$
$\mathrm{Pin}(3)$	pinory (?)	$4\|G\|$	$(A_1, A_2) \rightarrow A_1A_2$

- e.g. the chiral icosahedral group has 60 elements, encoded by 120 spinors, which form the binary icosahedral group
- together with the inversion/pseudoscalar I this gives 60 rotations and 60 rotoinversions, i.e. the full icosahedral group H_3 in 120 elements doubly covered by 240 pinors

Representations from Clifford multivector groups

- The usual picture of orthogonal transformations on an n-dimensional vector space is via $n \times n$ matrices acting on vectors, immediately making connections with representations = matrices satisfying the group multiplication laws.
- Easy to construct representations with (s)pinors in the 2^n-dimensional Clifford algebra as reshuffling components.
- Spinors leave the original n-dimensional vector space invariant, reshuffle the components of the vector.
- But can also consider various representation matrices acting on different subspaces of the Clifford algebra.

Representations from Clifford multivector groups – trivial, parity, rotation representations

- The scalar subspace is one-dimensional. $\tilde{R}1R = \tilde{R}R = 1$ gives the trivial representation, and likewise pinors A give the parity.
- The double-sided action $\tilde{R}xR$ of spinors R on a vector x in the n-dimensional vector space gives an $n \times n$-dimensional representation, which is just the usual rotation matrices.
- E.g. e_1e_2 acting on $x = x_1e_1 + x_2e_2 + x_3e_3$ gives $e_2e_1xe_1e_2 = -x_1e_1 - x_2e_2 + x_3e_3$ which could also be expressed as $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_1 \\ -x_2 \\ x_3 \end{pmatrix}$
- If the spinors were acting as $Rx\tilde{R}$ would give a potentially different representation.

Characters, their norm, and the Frobenius-Schur indicator

- Similarity transformed representations are also good representations, but are not fundamentally different: they are equivalent.
- So want a measure for a representation that is invariant under similarity transformations, e.g. the trace aka the character χ of a matrix
- A class function i.e. the same within a conjugacy class because of the cyclicity of the trace
- The character norm $||\chi||^2 := \frac{1}{|G|}\sum_{g\in G}|\chi(g)|^2$
- The Frobenius-Schur indicator $\nu := \frac{1}{|G|}\sum_{g\in G}\chi(g^2)$

Real representations of real, complex, and quaternionic type

- $||\chi||^2 = \frac{1}{|G|}\sum_{g\in G}|\chi(g)|^2 = 1$: representation of real type
- $||\chi||^2 = \frac{1}{|G|}\sum_{g\in G}|\chi(g)|^2 = 2$: representation of complex type
- $||\chi||^2 = \frac{1}{|G|}\sum_{g\in G}|\chi(g)|^2 = 4$: representation of quaternionic type
- Theorem: A complex representation is irreducible if and only if $||\chi||^2 = 1$.
- Theorem: A real representation is irreducible if and only if $\boxed{||\chi||^2 + \nu(\chi) = 2}$, e.g. $4-2=2$ or $1+1=2$.

Representations from Clifford multivector groups – 8×8 and 4×4 (whole algebra / even subalgebra)

- Rather than restricting oneself to the n-dimensional vector space, one can also define representations by $2^n \times 2^n$-matrices acting on the whole Clifford algebra, i.e. any element acting on an arbitrary elemtent, e.g. here 8×8.
- Likewise, one can define $2^{(n-1)} \times 2^{(n-1)}$-dimensional spinor representations as acting on the even subalgebra.
- 3D spinors have components in $(1, e_1e_2, e_2e_3, e_3e_1)$, multiplication with another spinor e.g. e_1e_2 will reshuffle these components $(e_1e_2, -1, -e_3e_1, e_2e_3)$
- This reshuffling can therefore be described by a 4×4-matrix.

4×4 – explicit example: A_1^3

- E.g. $\boxed{\pm e_1, \pm e_2, \pm e_3}$ give the 8 spinors $\boxed{\pm 1, \pm e_1 e_2, \pm e_2 e_3, \pm e_3 e_1}$, or $(\pm 1, 0, 0, 0)$ (8 permutations)
- $||\chi||^2 = 32/8 = 4$, $\nu = -2$ and $||\chi||^2 + \nu = 2$ i.e. real irreducible of quaternionic type

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

Character table of Q

Q	1	-1	$\pm e_1 e_2$	$\pm e_2 e_3$	$\pm e_3 e_1$
1	1	1	1	1	1
$1'$	1	1	-1	-1	1
$1''$	1	1	-1	1	-1
$1'''$	1	1	1	-1	-1
2	2	-2	0	0	0
4_H	4	-4	0	0	0

4×4 – explicit example: A_3

- As a set of vectors in 4D, they are $(\pm 1, 0, 0, 0)$ (8 permutations) , $\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)$ (16 permutations)
- Conjugacy classes: $1 \cdot 4^2 + 1 \cdot (-4)^2 + 6 \cdot 0^2 + 8 \cdot 2^2 + 8 \cdot (-2)^2 = 32 + 32 + 32 = 96$
- $||\chi||^2 = 96/24 = 4$, $\nu = -2$ and $||\chi||^2 + \nu = 2$ i.e. real irreducible of quaternionic type.

3×3 – explicit example: H_3

- Icosahedral spinors are

$$(\pm 1, 0, 0, 0) \text{ (8 permutations) }, \frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1) \text{ (16 permutations)}$$

$$\frac{1}{2}(0, \pm 1, \pm \sigma, \pm \tau) \text{ (96 even permutations) },$$

- E.g. the rotation matrices corresponding to $\alpha_1\alpha_2$ and $\alpha_2\alpha_3$ via $\tilde{R}xR$ are

$$\frac{1}{2}\begin{pmatrix} \tau & \tau - 1 & -1 \\ 1-\tau & -1 & -\tau \\ -1 & \tau & 1-\tau \end{pmatrix} \text{ and } \frac{1}{2}\begin{pmatrix} \tau & 1-\tau & -1 \\ 1-\tau & 1 & -\tau \\ 1 & \tau & \tau - 1 \end{pmatrix}.$$

  The characters $\chi(g)$ are obviously 0 and τ
- $||\chi||^2 = 120/120 = 1$, $\nu = 1$ and $||\chi||^2 + \nu = 2$ i.e. real irreducible of real type

3×3 – explicit example: H_3 other way

- If the spinors were acting as $Rx\tilde{R}$, then

$$\frac{1}{2}\begin{pmatrix} \tau & 1-\tau & -1 \\ \tau-1 & -1 & \tau \\ -1 & -\tau & 1-\tau \end{pmatrix} \text{ and } \frac{1}{2}\begin{pmatrix} \tau & 1-\tau & 1 \\ 1-\tau & 1 & \tau \\ -1 & -\tau & \tau-1 \end{pmatrix},$$

with the same characters as before. Swapping the action of the spinor can change the representation.

4×4 – explicit example: H_3

- Spinors $\alpha_1\alpha_2$ and $\alpha_2\alpha_3$ multiplying a generic spinor $R = a_0 + a_1 e_2 e_3 + a_2 e_3 e_1 + a_3 e_1 e_2$ from the left reshuffles the components (a_1, a_2, a_3, a_0) with the matrices given as

$$\frac{1}{2}\begin{pmatrix} -1 & \tau-1 & 0 & -\tau \\ 1-\tau & -1 & -\tau & 0 \\ 0 & \tau & -1 & \tau-1 \\ \tau & 0 & 1-\tau & -1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} -\tau & 0 & 1-\tau & -1 \\ 0 & -\tau & -1 & \tau-1 \\ \tau-1 & 1 & -\tau & 0 \\ 1 & 1-\tau & 0 & -\tau \end{pmatrix},$$

with characters -2 and -2τ.

4×4 – explicit example H_3: quaternionic type

- 120 4×4 matrices – 9 conjugacy classes, with pairs that have $\pm 2\chi_3$ so gives 4 times that of the 3×3 case
- $|G| \cdot ||\chi||^2 = 1 \cdot 4^2 + 1 \cdot (-4)^2 + 12 \cdot (-2\tau)^2 + 12 \cdot (2\tau)^2 + 12 \cdot (-2\sigma)^2 + 12 \cdot (2\sigma)^2 + 20 \cdot (-2)^2 + 20 \cdot (2)^2 + 30 \cdot 0^2 = 480$
- $||\chi||^2 = 480/120 = 4$, $\nu = -2$ and $||\chi||^2 + \nu = 2$ i.e. real irreducible of quaternionic type

Character table of $I = A_5$

I	1	$20C_3$	$15C_2$	$12C_5$	$12C_5^2$
1	1	1	1	1	1
3	3	0	-1	τ	σ
$\bar{3}$	3	0	-1	σ	τ
4	4	1	0	-1	-1
5	5	-1	1	0	0

Character table of $2I$

I	1	$20C_3$	$30C_2$	$12C_5$	$12C_5^2$	-1	$-20C_3$	$-12C_5$	$-12C_5^2$
1	1	1	1	1	1	1	1	1	1
3	3	0	-1	τ	σ	3	0	τ	σ
$\bar{3}$	3	0	-1	σ	τ	3	0	σ	τ
4	4	1	0	-1	-1	4	1	-1	-1
5	5	-1	1	0	0	5	-1	0	0
2	2	-1	0	$-\sigma$	$-\tau$	-2	1	σ	τ
2	2	-1	0	$-\tau$	$-\sigma$	-2	1	τ	σ
4	4	1	0	-1	-1	-4	-1	1	1
6	6	0	0	1	1	-6	0	-1	-1
4_H	4	-2	0	-2τ	-2σ	-4	2	2τ	2σ
$4_{\tilde{H}}$	4	-2	0	-2σ	-2τ	-4	2	2σ	2τ

A general construction of representations of quaternionic type – canonical representations

- It had so far been overlooked that there is a systematic construction of representations of quaternionic type for 3D polyhedral groups
- This is simply due to the fact that the spinors in 3D provide a realisation of the quaternions
- Therefore spinors provide 4x4 representations of quaternionic type for all (though limited number of) possible groups
- However, they are canonical for a choice of 3D simple roots, i.e. there is a preferred amongst all similarity transformed versions
- These simple roots also determine the 3x3 rotation matrices and their reversed representations in a similar canonical way

Characters in general

- For a general spinor $R = a_0 + a_1e_2e_3 + a_2e_3e_1 + a_3e_1e_2$ one has 3D character $\chi = 3a_0^2 - a_1^2 - a_2^2 - a_3^2$ and representation

$$\frac{1}{2}\begin{pmatrix} a_0^2+a_1^2-a_2^2-a_3^2 & -2a_0a_3+2a_1a_2 & 2a_0a_2+2a_1a_3 \\ 2a_0a_3+2a_1a_2 & a_0^2-a_1^2+a_2^2-a_3^2 & -2a_0a_1+2a_2a_3 \\ -2a_0a_2+2a_1a_3 & 2a_0a_1+2a_2a_3 & a_0^2-a_1^2-a_2^2+a_3^2 \end{pmatrix}$$

- and the 4D rep and character are

$$\begin{pmatrix} a_0 & a_3 & -a_2 & a_1 \\ -a_3 & a_0 & a_1 & a_2 \\ a_2 & -a_1 & a_0 & a_3 \\ -a_1 & -a_2 & -a_3 & a_0 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_0 \end{pmatrix} \text{ and } \chi = 4a_0.$$

- Characters of the representations are all determined by the spinor!

Overview

1. Polyhedral groups, Platonic solids and root systems
2. Reflection groups with Clifford algebras
   - A Clifford way of doing orthogonal transformations
   - The geometry of the Coxeter plane
   - Root system induction and ADE correspondences
   - Representations from multivector groups
   - Conformal, modular and braid groups
3. Conclusions

Clifford Algebra and orthogonal transformations

- Inner product is symmetric part $a \cdot b = \frac{1}{2}(ab + ba)$
- Reflecting a in b is given by $\boxed{a' = a - 2(a \cdot b)b = -bab}$ (b and $-b$ doubly cover the same reflection)
- Via Cartan-Dieudonné theorem any orthogonal (/conformal/modular) transformation can be written as successive reflections

$$\boxed{x' = \pm n_1 n_2 \dots n_k x n_k \dots n_2 n_1} = \pm A x \tilde{A}$$

- The conformal group $C(p, q) \sim SO(p + 1, q + 1)$ so can use these for translations, inversions etc as well

Conformal Geometric Algebra

- Go to $e_1, e_2, e, \bar{e}$, with $e_0^2 = 1, e_i^2 = -1, e^2 = 1, \bar{e}^2 = -1$
- Define two null vectors $n \equiv e + \bar{e}$, $\bar{n} \equiv e - \bar{e}$
- Can embed the 2D vector $x = x^\mu e_\mu = xe_1 + ye_2$ as a null vector in 4D (also normalise $F(x) \cdot e = -1$)

$$F(x) = \frac{1}{\lambda^2 - x^2}(x^2 n + 2\lambda x - \lambda^2 \bar{n})$$

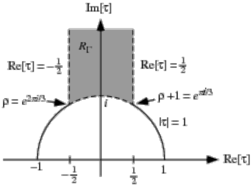
- So neat thing is that conformal transformations are now done by rotors (except inversion which is a reflection) – distances are given by inner products

Conformal Transformations in CGA

$$F(x) = \frac{1}{\lambda^2 - x^2}(x^2 n + 2\lambda x - \lambda^2 \bar{n})$$

- Reflection: spacetime $F(-axa) = -aF(x)a$
- Rotation: spacetime $F(Rx\tilde{R}) = RF(x)\tilde{R}$, $R = \exp(\frac{ab}{2\lambda})$
- Translation: $F(x+a) = R_T F(x)\tilde{R}_T$ for $R_T = \exp(\frac{na}{2\lambda}) = 1 + \frac{na}{2\lambda}$
- Dilation: $F(e^\alpha x) = R_D F(x)\tilde{R}_D$ for $R_D = \exp(\frac{\alpha}{2\lambda} e\bar{e})$
- Inversion: Reflection in extra dimension e: $F(\frac{x}{x^2}) = -eF(x)e$ sends $n \leftrightarrow \bar{n}$
- Special conformal transformation: $F(\frac{x}{1+ax}) = R_S F(x)\tilde{R}_S$ for $R_S = R_I R_T R_I$

Modular group



- Modular generators: $T : \tau \to \tau + 1$, $S : \tau \to -1/\tau$
- $\boxed{\langle S, T | S^2 = I, (ST)^3 = I \rangle}$ CGA rotor version: $\boxed{R_Y X \tilde{R}_Y}$
- CGA: $\boxed{T_X = \exp(\frac{ne_1}{2}) = 1 + \frac{ne_1}{2}}$ and $\boxed{S_X = e_1 e}$ (slight issue of complex structure $\tau =$ complex number, not vector in the 2D real plane so map $e_1 : x_1 e_1 + x_2 e_2 \leftrightarrow x_1 + x_2 e_1 e_2 = x_1 + i x_2$)
- $\boxed{(S_X T_X)^3 = -1}$ and $\boxed{S_X^2 = -1}$
- So a 3-fold and a 2-fold rotation in conformal space

Braid group

- $(S_X T_X)^3 = -1$ and $S_X^2 = -1$ is inherently spinorial
- Of course Clifford construction gives a double cover
- The braid group is a double cover
- So Clifford construction gives the braid group double cover of the modular group
- $\sigma_1 = \tilde{T}_X = \exp(-ne_1/2)$ and $\sigma_2 = T_X S_X T_X = \exp(-\bar{n}e_1/2)$ satisfying $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ $(= S_X)$
- Nice symmetry between the roles of the point at infinity and the origin
- Might not be known? Spinorial techniques might make awkward modular transformations more tractable?

Conclusions

- Clifford algebra provides a very general way of doing reflection group theory (Cartan-Dieudonné)
- Construction of the exceptional root systems from 3D root systems
- More geometric approach to the geometry of the Coxeter plane, degrees and exponents
- Geometry of 3D space systematically and canonically gives representations of 4D root systems in terms of quaternions and polyhedral representations of quaternionic type (among others)

Conclusions

Thank you!

Quaternion groups via the geometric product

- The 8 quaternions of the form $(\pm 1, 0, 0, 0)$ and permutations are the Lipschitz units,the quaternion group in 8 elements.
- The 8 Lipschitz units together with $\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)$ are the Hurwitz units, the binary tetrahedral group of order 24. Together with the 24 'dual' quaternions of the form $\frac{1}{\sqrt{2}}(\pm 1, \pm 1, 0, 0)$, they form the binary octahedral group of order 48.
- The 24 Hurwitz units together with the 96 unit quaternions of the form $(0, \pm\tau, \pm 1, \pm\sigma)$ and even permutations, are called the Icosians. The icosian group is isomorphic to the binary icosahedral group with 120 elements.
- The unit spinors $\{1; e_2e_3; e_3e_1; e_1e_2\}$ of $\mathrm{Cl}(3)$ are isomorphic to the quaternion algebra $\mathbb{H}$.

H_4 from icosahedral spinors

- The H_3 root system has 30 roots e.g. simple roots $\alpha_1 = e_2, \alpha_2 = -\frac{1}{2}((\tau-1)e_1 + e_2 + \tau e_3)$ and $\alpha_3 = e_3$.
- The subgroup of rotations is A_5 of order 60
- These are doubly covered by 120 spinors of the form $\alpha_1\alpha_2 = -\frac{1}{2}(1-(\tau-1)e_1e_2 + \tau e_2e_3)$, $\alpha_1\alpha_3 = e_2e_3$ and $\alpha_2\alpha_3 = -\frac{1}{2}(\tau-(\tau-1)e_3e_1 + e_2e_3)$.
- As a set of vectors in 4D, they are

$$(\pm 1, 0, 0, 0) \text{ (8 permutations)}, \quad \frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1) \text{ (16 permutations)}$$

$$\frac{1}{2}(0, \pm 1, \pm\sigma, \pm\tau) \text{ (96 even permutations)},$$

which are precisely the 120 roots of the H_4 root system.

Systematic construction of the polyhedral groups

- Multiplying together root vectors in the Clifford algebra gave a systematic way of constructing the binary polyhedral groups as 3D spinors $=$ quaternions.
- The 6/12/18/30 roots in $A_1 \times A_1 \times A_1 / A_3 / B_3 / H_3$ generate 8/24/48/120 spinors.
- The discrete spinor group is isomorphic to the quaternion group Q / binary tetrahedral group $2T$/ binary octahedral group $2O$/ binary icosahedral group $2I$).

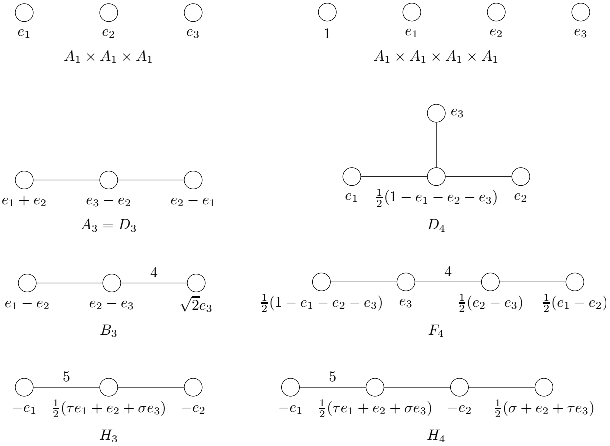
A_1^3 A_3 B_3 H_3

A_1^4 D_4 F_4 H_4

Quaternionic representations of 3D and 4D Coxeter groups

- Groups E_8, D_4, F_4 and H_4 have representations in terms of quaternions
- Extensively used in the high energy physics/quasicrystal/Coxeter/polytope literature and thought of as deeply significant, though not really clear why
- e.g. H_4 consists of 120 elements of the form $(\pm 1, 0, 0, 0)$, $\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)$ and $(0, \pm\tau, \pm 1, \pm\sigma)$
- Seen as remarkable that the subset of the 30 pure quaternions is a realisation of H_3 (a sub-root system)
- Similarly, B_3 and $A_1 \times A_1 \times A_1$ have representations in terms of pure quaternions
- Clifford provides a much simpler geometric explanation

Quaternionic representations in the literature



Pure quaternions = Hodge dualised root vectors

Quaternions = spinors

Demystifying Quaternionic Representations

- Pure quaternion subset of 4D groups only gives 3D group if the 3D group contains the inversion/pseudoscalar I
- e.g. does not work for the tetrahedral group A_3, but $A_3 \to D_4$ induction still works, with the central node essentially 'spinorial'
- In fact, it goes the other way around: the 3D groups induce the 4D groups via spinors
- The rank-4 groups are also generated (under quaternion multiplication) by two quaternions we can identify as $R_1 = \alpha_1\alpha_2$ and $R_2 = \alpha_2\alpha_3$
- Can see these are 'spinor generators' and how they don't really contain any more information/roots than the rank-3 groups alone

Quaternions vs Clifford versors

- Sandwiching is often seen as particularly nice feature of the quaternions giving rotations
- This is actually a general feature of Clifford algebras/versors in any dimension; the isomorphism to the quaternions is accidental to 3D
- However, the root system construction does not necessarily generalise
- 2D generalisation merely gives that $I_2(n)$ is self-dual
- Octonionic generalisation just induces two copies of the above 4D root systems, e.g. $A_3 \to D_4 \oplus D_4$
- Recently constructed E_8 from the 240 pinors doubly covering 120 elements of H_3 in $2^3 = 8$-dimensional 3D Clifford algebra